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# 1/2-Order Fractional Fokker–Planck Equation on Combike Model

**M. A. Zahran**[1]

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From the generalized scheme of random walks on the comblike structure, it is shown how a 1/2-order fractional Fokker–Planck equation can be derived. The operator method for the moments associated with the distribution function $p(x, t)$ is used to solve the resulting equation. Also the anomalous diffusion along the backbone of the structure has been considered.

**KEY WORDS:** Anomalous diffusion; fractional Fokker–Planck equation; operator method.

## 1. INTRODUCTION

As is known, anomalous diffusion characterizing via the mean square displacement,

$$\langle X^2(t)\rangle \sim t^\beta, \qquad \beta = 2/(2+\theta), \tag{1}$$

is deviating from the normal diffusion behavior; $\beta = 1$. Such deviation has for instance been observed for transport, including charge transport in amorphous semiconductor,[1–3] the dynamics of a bead in a polymer network,[4] and diffusion of water in biological tissues[5] or, more generally, in the disordered systems.[6]

According to the value of the anomalous diffusion exponent $\beta$, one can distinguish between superdiffusion domain ($\beta > 1$), such as the anomalous diffusion of adsorbed molecules,[7] and subdiffusion domain ($\beta < 1$), such as diffusion on fractals.

[1] Physics Department, Faculty of Science, Mansoura University, Mansoura, Egypt; e-mail: m_zahran1@mans.edu.eg

On fractals supports, the change of the diffusion character is due to the strong tortuous nature of ways (twistedness) and presence of impasses, i.e., dead ends on current ways or the lacunarity of the structure, that is, the presence of holes of all length scales.[6] In the following, we mainly deal with the subdiffusion domain.

Normal diffusion under the influence of an external force field is often modeled by one dimension Fokker–Planck equation (FPE)

$$\partial p(x,t)/\partial t = L_{\mathrm{FP}}\, p(x,t), \tag{2}$$

where $L_{\mathrm{FP}}$ is the Fokker Planck operator given by,

$$L_{\mathrm{FP}} = \{\partial^2/\partial x^2[D(x)] - \partial/\partial x[V(x)]\}. \tag{3}$$

In Eq. (2), $p(x,t)$ is the probability density for the position $x$ of the diffusing particle at time $t$. $D(x)$ and $V(x)$ are coefficients associated with the diffusion and external drift respectively. Usually, Eq. (2) can be derived following the Langevin approach, that is starting from the stochastic equation of motion for the dynamical variable whose probability distribution we are interested in.[8,9] The basic properties of FPE are the exponential decay of the single modes in time,

$$T_n(t) = \exp(-\lambda_{n,1} t), \tag{4}$$

where $\lambda_{n,1}$ is the eigenvalue of the operator $L_{\mathrm{FP}}$. In the absence of an external drift term, i.e., $V(x)=0$, the equation describes a Gaussian evolution as may be anticipated based on the central limit theorem. Also the square-mean displacement $\langle X^2(t)\rangle$ is proportional to time $t$, i.e., $\beta = 1$.

In analogy to the description of normal diffusion in an external field via the FPE, one may expect to obtain FPE involved with a fractional integral operator $\partial_t^{-\beta} p(x,t)$, $(0<\beta<1)$, to model anomalous diffusion under the influence of an external field. It is worth mentioning that fractional kinetic equations have been extensively shown to be a well-suited tool for the description of anomalous diffusion [ref. 10 and refs. therein; refs. 11–13]. Here we demonstrate the derivation of fractional Fokker–Planck equation (FFPE) with $\beta = 1/2$ by using the comblike model as an extension to ref. 14. The comb structure was put forward in refs. 15–19, and as shown in Fig. 1, it consists of a main channel or backbone along the $x$-axis where the random walker diffusively connects with an infinite length of fingers. Using the technique of the generating functions, it was shown that the mean-square displacement along the axis of structure depends on time in the anomalous way (1) with the exponent $\theta = 2$; i.e., $\beta = 1/2$. So, the main purpose of the present paper is to derive the FFPE which

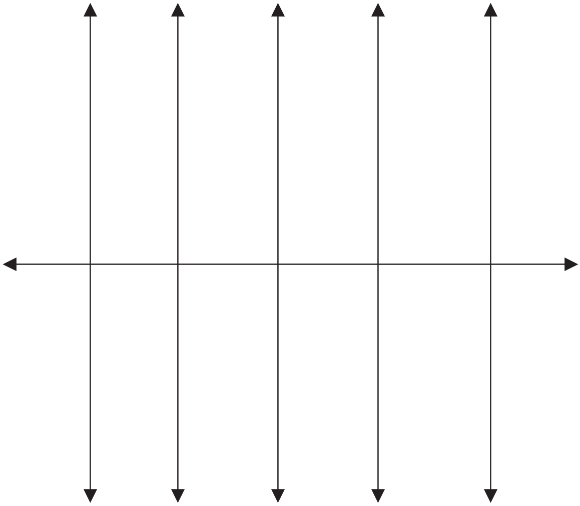

Fig. 1. Comb structure: the conducting axis $y = 0$ has fingers going to infinity.

essentially differs from the usual FPE, i.e., instead of the first order derivative of time, a derivative of the fractional order $\beta = 1/2$ arises,

$$p(x, t) = {}_0D_t^{-1/2} L^*_{\mathrm{FP}} p(x, t), \tag{5}$$

with

$$L^*_{\mathrm{FP}} = \partial^2/\partial x^2[D^*(x)] - \partial/\partial x[V^*(x)] \tag{6}$$

where

$${}_0D_t^{-1/2} p(x, t) = 1/\Gamma(1/2) \int_0^t dt' \; p(x, t')/(t - t')^{1/2}, \tag{7}$$

is the Liouvell–Riemann integral operator.[20] $D^*(x)$ and $V^*(x)$ are referred to as the diffusion and drift coefficients along the main axis of the comblike structure under consideration. $\Gamma(m)$ is the Gamma function.

The paper is organized as follows: in Section 2 we use a comblike model to derive the FFPE with 1/2 order time evolution operator. Section 3 is devoted to solve the resulting equation via the operator method. The anomalous diffusion along the axis of structure is considered in the last section.

## 2. 1/2-ORDER FFPE ON COMB STRUCTURE

To derive the FFPE, we recall the usual description introduced in ref. 14. Consider the $x$-component of the current change along the axis of structure ($y = 0$),

$$J_x = \delta(y)[V(x) - \partial/\partial x\, D(x)]\, p(x, y, t), \tag{8}$$

$D(x)$ and $V(x)$ are assumed to be the diffusion coefficient and the potential acting on the diffusion particle along the $x$-axis, respectively. The $y$-component of the current along the fingers is given by

$$J_y = -D_y\, \partial p(x, y, t)/\partial y, \tag{9}$$

$D_y$ is the diffusion coefficient along the fingers and assumed to be constant. From the continuity equation,

$$\partial p(x, y, t)/\partial t = -[\partial J_x/\partial x + \partial J_y/\partial y], \tag{10}$$

one can write the corresponding diffusion-drift equation as,

$$[\partial/\partial t - \delta(y)\, \partial/\partial x(\partial/\partial x\, D(x) - V(x)) - D_y\, \partial^2/\partial y^2]\, p(x, y, t) = 0, \tag{11}$$

where $p(x, y, t)$ is the concentration of the particles on the comb structure. Equation (11) can be rewritten as a normal diffusion along the fingers with a non-uniform right term,

$$[\partial/\partial_t - D_y\, \partial^2/\partial y^2]\, p(x, y, t) = \delta(y)\, \partial/\partial x(\partial/\partial x\, D(x) - V(x)). \tag{12}$$

It is known that the homogenous part of the above Eq. (12) admits the Gaussian distribution,

$$G(y,t) = \exp - (y^2/4t)/(\pi D_y t)^{1/2}. \tag{13}$$

Employing the integration over the source term, we receive the integral equation,

$$p(x, y, t) = \int dy'\, dt'\, G(y - y', t - t')\, \delta(y')\, \partial/\partial x(\partial/\partial x\, D(x) - V(x))\, p(x, y', t'). \tag{14}$$

Equation (14), upon integration over $y'$, gives the closed equation for the concentration of the particles along the main channel,

$$p(x, t) = \partial/\partial x(\partial/\partial x\, D^*(x) - V^*(x))\, {}_0D_t^{-1/2}\, p(x, t), \tag{15}$$

under the following consideration,

$$D^*(x) = D(x)/D_y, \qquad V^*(x) = V(x)/D_y. \tag{16}$$

It will become obvious that Eq. (15) represents the fractional FPE which is recently proposed and discussed in ref. 10. Assuming that the diffusion coefficient and the external drift are given by $D^*(x) = (x^{1-2\varepsilon}/4)$, and $V^*(x) = (x^{-2\varepsilon}/4)$ respectively, where $\varepsilon$ is an arbitrary parameter that leads to different types of FFPE. Therefore, the FFPE takes the general form

$$p(x, t) = {}_0D_t^{-1/2}[\partial^2/\partial x^2(x^{1-2\varepsilon}/4) - \partial/\partial x(x^{-2\varepsilon}/4)]\, p(x, t). \tag{17}$$

Making use of the replacements,[21]

$$\begin{aligned} p(x, t) &= (2)^{1/2}\, x^{(2\varepsilon-1)/2} w(z, \tau), \\ \tau &= t, \\ z &= [2(2)^{1/2}/2\varepsilon+1]\, x^{(2\varepsilon+1)/2}, \end{aligned} \tag{18}$$

under the constraint that $\varepsilon \neq -1/2$ (we remark that at $\varepsilon = -1/2$, this problem can be treated directly from Eq. (17) with the same technique introduced later), Eq. (17) reduces to the fractional FPE associated with fractional Brownian motion as,

$$w(z, \tau) = {}_0D_\tau^{-1/2} \bar{L}_{\mathrm{FP}} w(z, \tau), \tag{19}$$

where $\bar{L}_{\mathrm{FP}}$ is given by,

$$\bar{L}_{\mathrm{FP}} = \partial^2/\partial z^2(w(z, \tau)/2) - \partial/\partial z(w(z, \tau)/2z). \tag{20}$$

In order to get a formal solution for Eq. (19), we use the Laplace–Mellin technique[22] as follows: we define,

$$w(z, u) = L(w(z, \tau), u) = \int_0^\infty d\tau \exp(-u\tau)\, w(z, \tau), \tag{21}$$

as the Laplace transform and,

$$w(z, s) = M(w(z, \tau), s) = \int_0^\infty d\tau\, \tau^{s-1} w(z, \tau), \tag{22}$$

as the Mellin transform of $w(z, \tau)$. The connection between Laplace and Mellin transforms is defined by,

$$w(z, s) = [1/\Gamma(1-s)]\; M(L(w(z, \tau), u), 1-s). \tag{23}$$

Hence, taking Eq. (19) into the Laplace domain yields,

$$w(z, u) = u^{-1/2}(u^{1/2} - \bar{L}_{\mathrm{FP}})^{-1}\, w(z, 0), \tag{24}$$

assuming that $w(z, 0)$ is the initial condition. Inserting Eq. (24) into Eq. (23) one finds,

$$w(z, s) = (2\Gamma(2s)\,\Gamma(1-2s)/\Gamma(1-s))(\bar{L}_{\mathrm{FP}})^{-2s}\, w(z, 0). \tag{25}$$

Comparing the inverse of Mellin transform with the definition of $H$-function[23, 24] gives,

$$w(z, \tau) = 2H^{11}_{12}\big((\bar{L}_{\mathrm{FP}})^2\, \tau\big|^{(0,2)}_{(0,2)(0,1)}\big)\, w(z, 0). \tag{26}$$

Here,

$$H^{mn}_{pq}(Z) = H^{mn}_{pq}\big(Z\big|^{(a_j,\alpha_j)_{j=1\cdots p}}_{(b_j,\beta_j)_{j=1\cdots q}}\big), \tag{27}$$

denotes the $H$-function, and can be represented by contour integration as,

$$H^{mn}_{pq}(Z) = (1/2\pi i)\int ds(A(s)\,B(s))/C(s)\,D(s), \tag{28}$$

with,

$$\begin{aligned} A(s) &= \prod_{j=1}^{m} \Gamma(b_j - \beta_j s),\\ B(s) &= \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s),\\ C(s) &= \prod_{j=1+m}^{q} \Gamma(1 - b_j - \beta_j s), \end{aligned} \tag{29}$$

and

$$D(s) = \prod_{j=1+n}^{p} \Gamma(a_j - \alpha_j s).$$

The $H$-function can be written in a series expansion such as

$$w(z,\tau)=\left\{\sum_{r=0}^{\infty}\frac{(\bar{L}_{\mathrm{FP}})^{r}\,\tau^{1/2}}{\Gamma((r/2)+1)}\right\}w(z,0), \tag{30}$$

or

$$w(z,\tau)=E_{1/2}(\bar{L}_{\mathrm{FP}}\tau^{1/2})\,w(z,0), \tag{31}$$

$E_{\beta}(Z)$ denotes the Mittag–Leffler function and the series representation for it is given by,[23]

$$E_{\beta}(Z)=\left\{\sum_{r=0}^{\infty}\frac{Z^{r}}{\Gamma(\beta r+1)}\right\}. \tag{32}$$

Evidently the solution (31) represents very slow non-exponential dynamics approaching algebraic time decay in the long limit. When the order of FFPE tends to the usual order of time evolution operator, it is easy to show that Eq. (31) reduces to the exponential formal solution,

$$w(z,\tau)=\exp(\bar{L}_{\mathrm{FP}}\tau)\,w(z,0). \tag{33}$$

Also the basic properties of FFPE are the monotonically decreasing Mittag–Leffler function of the single modes in time,

$$T_{n}(\tau)=E_{1/2}(-\lambda_{n,1/2}\tau^{1/2}). \tag{34}$$

Comparing with exponential decay of Eq. (4), one can show that Eq. (34) represents an exact relaxation function for an underlying fractal time walk process, and that function directly leads to the cole–cole behavior[25] for the complex susceptibility, which is widely used to describe the experimental results.

## 3. SOLUTION OF FFPE VIA OPERATOR METHOD

Consider the FFPE Eq. (19),

$$w(z,\tau)={}_{0}D_{\tau}^{-1/2}[\partial^{2}/\partial z^{2}(w(z,\tau)/2)-\partial/\partial z(w(z,\tau)/2z)]$$

with the boundary condition,

$$\lim_{z\to\pm\infty} z^{k}w(z,\tau)=0 \tag{35}$$

and initial condition,

$$w(z,0)=\delta(z-z_0), \qquad 0<z_0<1. \tag{36}$$

Multiplying Eq. (19) by $z^k$ and integrating over $z\in\{-\infty,\infty\}$, the corresponding moment equation takes the form,

$$M_k(\tau)=(1/2)\ {}_0D_\tau^{-1/2}\{kM_{k-2}(\tau)+k(k-1)\ M_{k-2}(\tau)\}, \tag{37}$$

where

$$M_k(\tau)=\int_{-\infty}^{\infty} dz\ z^k w(z,\tau). \tag{38}$$

Equation (37) can be rewritten in the operator form,

$$M_k(\tau)={}_0D_\tau^{-1/2}\bar{C}M_k(\tau). \tag{39}$$

The operator $\bar{C}$ is defined as,

$$\bar{C}=(1/2)(k+k(k-1))\ E^{-2} \tag{40}$$

where $E^{\pm\sigma}$ is the translation operator defined through,[26]

$$E^{\pm\sigma}M_k(\tau)=M_{k\pm\sigma}(\tau). \tag{41}$$

The general solution of Eq. (39) through Laplace–Mellin transforms becomes,

$$M_k(\tau)=\sum_{r=0}^{\infty}\frac{\tau^{r/2}(1/2)^r\ (k+k(k-1))^r}{\Gamma((r/2)+1)}M_{k-2r}. \tag{42}$$

The advantage of applying the operator method is obtaining a closed form for different order of the moments which can be directly evaluated via Eq. (42). For example, when $k=1,2$, the first two moments are really found to be,

$$M_1(\tau)\sim c_1(k)\ \tau^{1/2}/\Gamma(3/2), \tag{43a}$$

and

$$M_2(\tau)\sim c_2(k)\ \tau^{1/2}/\Gamma(3/2)+c_3(k)\ \tau/\Gamma(2), \tag{43b}$$

which increases sub-linearly in time and is very similar to the power law that describes the mean-square displacement, which characterizes the anomalous diffusion. $c_i(k)$ depends upon the order of the moments. In the case of first order time evolution, one obtains

$$M_k(\tau) = \exp(\tau \bar{C})\, M_k(0), \tag{44}$$

and the corresponding moments relation

$$M_k(\tau) = \sum_{r=0}^{\infty} \frac{\tau^r (1/2)^r\, (k + k(k-1))^r}{r!} M_{k-2r}. \tag{45}$$

## 4. ANOMALOUS DIFFUSION ON COMB STRUCTURE

Anomalous diffusion on the axis of structure for a comb model takes the form,[14]

$${}_0D_t^{1/2}\, p(x,t) = D\, \partial^2/\partial x^2\, p(x,t). \tag{46}$$

We can make a little advanced step by generalizing the order of differential operator with $\beta$, $0 < \beta < 1$, which yields,[22]

$${}_0D_t^{\beta}\, p(x,t) = D\, \partial^2/\partial x^2\, p(x,t). \tag{47}$$

Taking Eq. (47) into the Laplace–Fourier domain this equation becomes,

$$p(K,u) = u^{\beta+1}/(DK^2 + u^{\beta}), \tag{48}$$

For a well-behaved function, the mean-square displacement follows the form,

$$\langle X^2(t)\rangle = \int dx\, x^2 p(x,t) \equiv -\partial^2/\partial K^2\, p(K,u)|_{K=0}, \tag{49}$$

where the right-hand side may be found by expansion in $K^2$. Now, using the expression (48) into (49), one finds

$$\langle X^2(t)\rangle \sim 1/u^{\beta+1} \tag{50}$$

which is equivalent to Eq. (1) in time domain. Fourier inverse for Eq. (48) gives

$$p(x,u) = (x/2\pi D)^{1/2}\, u^{(3\beta/4)-1} K_{-1/2}(x u^{\beta/2}/D^{1/2}), \tag{51}$$

$K_\gamma$ denotes the modified Bessel function of the second kind. Applying the Laplace–Mellin connection to the above expression, one obtains,

$$p(x, s) = [(x/2\pi D)^{1/2}/\Gamma(1-s)] \int_0^\infty du\, u^{(3\beta/4)-1-s} K_{-1/2}(xu^{\beta/2}/D^{1/2}). \quad (52)$$

Upon using the integration procedure in ref. 23, we have,

$$p(x, t) = \pi^{1/2} \times H_{12}^{20}((x^2/4Dt^\beta)|_{(\beta, 1)(1, 1)}^{(1, \beta)}). \quad (53)$$

The propagator $p(x, t)$ corresponding to the anomalous diffusion exponent $\beta = 1/2$ along the axis of the comblike model is shown in Figs. 2 and 3 for different values of time $t$.

## 5. CONCLUSION

Although there has been considerable interest in the problem of anomalous diffusion on fractal structure, most results are known only from

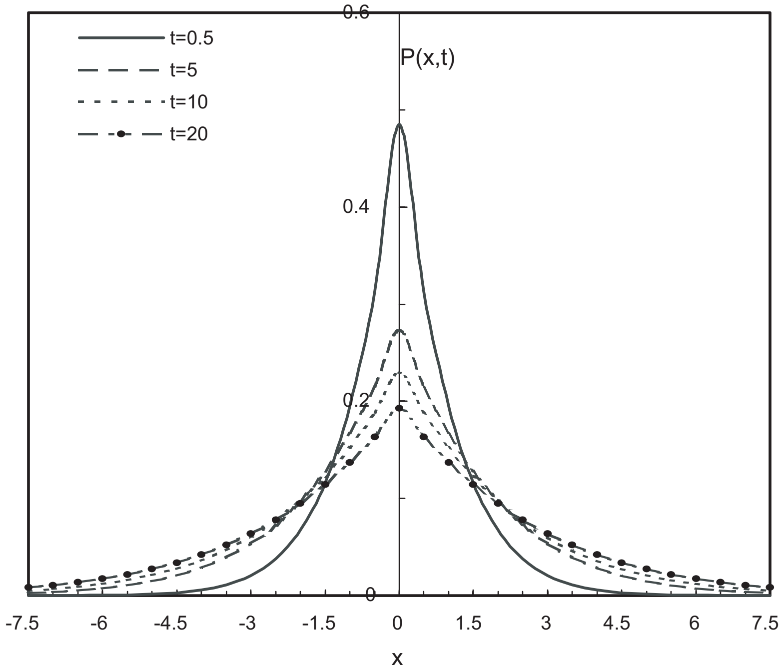


Fig. 2. The distribution function $p(x, t)$ along the axis of structure for the consecutive times $t = 0.5, 5, 10, 20$ in two dimensions.

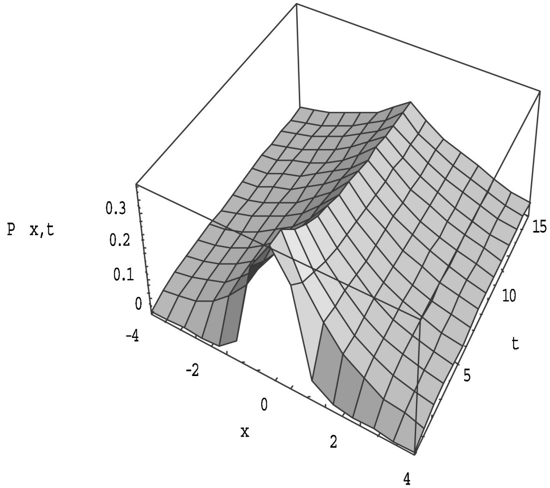


Fig. 3. The behavior of the distribution function $p(x, t)$ with respect to the space and time along the axis of comblike structure in three-dimensional diagram.

numerical simulations or scaling theories. The comblike model, inspite of its apparent simplicity, is one of the models in which one can obtain the exact solution of the anomalous diffusion problem. In this work, the comblike structure as a model for disordered systems is used to derive 1/2-order FFPE. The resulting equation has been recently proposed in the literature to describe the anomalous diffusion under the effect of external fields. Here, we introduce a technique, namely the operator method, to cast the FFPE into the fractional moment equation. The different orders of moments can be obtained via Laplace–Mellin transforms and the results show a sub-linear increase in time. In a force-free trapping-walk case, the mean square displacement is similar to the hall-mark law that characterizes the anomalous diffusion processes. This model offers some physical insights into the origin of fractional dynamics for a system which exhibits multiple trapping such as the charge carrier transport in amorphous semiconductors,[2, 27] or the phase space dynamics of chaotic Hamiltonian systems.[28] Also, experimental realization might be found in porous media or possible in Gel Electrophoresis.[29]

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